HOW TO ASSIGN PROBABILITIES IF YOU MUST ADDITION ON ARTICLE

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ABSTRACT. In the article 'How To Assign Probabilities if you must'[1] several methods to assign probabilities applied to a two-played die rolling game were discussed. The main focus was on methods using the logarithmic loss function for the choice of proper loss function. In this additional part, we will discuss the Brier and Epstein loss functions as an alternative. Futhermore, an extensive graphical display of the situation will be made. For notation and terminology, the reader is referred to Albers *et al.*[1]

1. EXTENSION TO OTHER PROPER LOSS FUNCTIONS

We consider three proper loss functions: logarithmic ($L_{\log}$), Brier (L_B) and Epstein (L_E) loss

$$L_{\log}(y, Q(x)) = -\log q_x(y) \tag{1}$$

$$L_B(y, Q(x)) = (1 - q_x(y))^2 + \sum_{\eta \neq y} (q_x(\eta))^2 \tag{2}$$

$$L_E(y, Q(x)) = \sum_{\eta=1}^{6} \left(\mathbf{1}_{\{y,\dots,6\}}(\eta) - \sum_{\nu=1}^{\eta} q_x(\nu) \right)^2 \tag{3}$$

For the seven possible combinations of x and y, the losses incorporated are:

x	y	logarithmic	Brier	Epstein
1	1	$\log(2)$	0.5	1
1	5	$\log(2)$	0.5	1
2	2	$-\log(a)$	$6a^2 - 6a + 2$	$10a^2 - 12a + 4$
2	4	$-\log(a)$	$6a^2 - 6a + 2$	$10a^2 - 8a + 2$
2	6	$-\log(1 - 2a)$	$6a^2$	$10a^2$
3	3	$-\log(b)$	$2(1 - b)^2$	$3(1 - b)^2$
3	6	$-\log(1 - b)$	$2b^2$	$3b^2$

From these losses, we can compute the risk functions $R_\cdot(0, Q_{a,b})$ and $R_\cdot(1, Q_{a,b})$ for Strategies 0 and 1 respectively, as well as the risk functions $R_\cdot(\theta, Q_{a,b})$ corresponding to the randomized strategy.

$$R_{\log}(\theta, Q_{a,b}) = -\tfrac{1}{6} \left[\log(2^{-2}a^2 b) - \theta \log(1 - 2a) - (1 - \theta) \log(1 - b) \right] \tag{4}$$

$$R_B(\theta, Q_{a,b}) = \tfrac{1}{6} \left[6(2 + \theta)a^2 - 12a + 2(2 - \theta)b^2 - 4b + 7 \right] \tag{5}$$

$$R_E(\theta, Q_{a,b}) = \tfrac{1}{6} \left[10(2 + \theta)a^2 - 20a + 3(2 - \theta)b^2 - 6b + 11 \right] \tag{6}$$

Due to the linear relation $P_\theta = (1-\theta)P_0 + \theta P_1$ and properness of the loss functions, the different envelope risks are obtained when the same procedures are used, as was already noted in the article. These procedures are of the form $Q_\theta = Q_{(2+\theta)^{-1},(2-\theta)^{-1}}$

with endpoints $Q_{\frac{1}{2},\frac{1}{2}}$ and $Q_{\frac{1}{3},1}$ if $\theta = 0, 1$. The envelope risk functions are

$$(7) \quad R^*_{\log}(\theta, Q_\theta) = -\tfrac{1}{6}[-\log 2^2 - [(2+\theta)\log(2+\theta) + (2-\theta)\log(2-\theta)] + [\theta\log\theta + (1-\theta)\log(1-\theta)]]$$

$$(8) \quad R^*_B(\theta, Q_\theta) = \frac{1}{6}\left[7 + \frac{1}{2+\theta} + \frac{5}{2-\theta} - \frac{28}{(2+\theta)(2-\theta)}\right]$$

$$(9) \quad R^*_E(\theta, Q_\theta) = \frac{1}{6}\left[11 + \frac{1}{2+\theta} + \frac{8}{2-\theta} - \frac{44}{(2+\theta)(2-\theta)}\right]$$

The shortcomings, or regrets, are obtained by substracting the envelope risks from the risks. Thus, $S(\theta, Q_\rho) = R(\theta, Q_\rho) - R(\theta, Q_\theta)$ for $\theta, \rho \in [0, 1]$. The regret functions for our three different loss functions are

$$S_{\log}(\theta, Q_\rho) = -\tfrac{1}{6}[-\theta\log\tfrac{\rho}{(2+\theta)^3(2-\theta)} + (1-\theta)\log\tfrac{1-\rho}{(2+\theta)^2(2-\theta)^2} + \theta\log\theta + (1-\theta)\log(1-\theta) - (2-\theta)\log(2-\theta) - (2+\theta)\log(2+\theta)]$$

$$S_B(\theta, Q_\rho) = \tfrac{(\rho-\theta)^2}{6(2+\rho)^2(2-\rho)^2(2+\theta)(2-\theta)}[(8-2\theta)\rho^2 + (-16+2\theta)\rho + (16-4\theta)]$$

$$S_E(\theta, Q_\rho) = \tfrac{(\rho-\theta)^2}{6(2-\rho)^2(2+\rho)^2(2-\theta)(2+\theta)}[(26-7\theta)\rho^2 + (-56+52\theta)\rho + (104-28\theta)]$$

These formulas are sufficient material for a graphical analysis of the procedures following from the three different loss functions. Our main interest goes out to the five 'special' actions, A ... E, as motivated in the article and specified in Table 1. In Table 2, for each of these points the corresponding procedure Q and regret points $S(0, Q), S(1, Q)$ are given (for all three loss functions).

A graphical representation can be seen in Figure 1 where, from above to below, two graphs for logarithmic, Brier and Epstein loss are displayed. The five points, along with the curve $\{Q_\rho, \rho \in [0, 1]\}$ are plotted. The three left graphs are plotted in the (a, b)-plane, where the rectangle bounds the procedures that deserve consideration. The three right graphs are plotted in the $(S(0, Q), S(1, Q))$-plane, the rectangle is now mapped 'into curves'.

In Figure 2 we have made a display of the risk for the case of logarithmic loss. Figure 3 shows a similar display, this time of the regret. For Brier and Epstein loss similar visualizations can be made, but we will not do so, as they don't have much additive value in this discussion. The left part of Figure 2 shows us the envelope

notation	description of action
A	Obtained by using simply the naive conditional probabilities.
B	Minimax regret procedure following from $S.(0, Q_\rho) = S.(1, Q_\rho)$.
C	The Bayes action w.r.t. a uniform prior ($Q_{\frac{1}{2}}$).
D	Minimax risk procedure following from $R.(0, Q_\rho) = R.(1, Q_\rho)$.
E	Naive action obtained by averaging the parameters of Q_0 and Q_1.

TABLE 1. Five actions that are of special interest.

	Logarithmic			Brier			Epstein		
	Q	$\theta = 0$	$\theta = 1$	Q	$\theta = 0$	$\theta = 1$	Q	$\theta = 0$	$\theta = 1$
A	$Q_{\frac{1}{3},\frac{1}{2}}$	.135	.116	$Q_{\frac{1}{3},\frac{1}{2}}$	.0556	.0833	$Q_{\frac{1}{3},\frac{1}{2}}$	.0926	.1250
B	$Q_{.4957}$	.093	.093	$Q_{.5359}$	.0447	.0447	$Q_{.5323}$	.0697	.0697
C	$Q_{\frac{1}{2}}$	.094	.092	$Q_{\frac{1}{2}}$	.0385	.0504	$Q_{\frac{1}{2}}$	.0611	.0778
D	$Q_{\frac{2}{3}}$	.144	.057	$Q_{.5359}$	.0447	.0447	$Q_{.5844}$	.0852	.0575
E	$Q_{\frac{5}{12},\frac{3}{4}}$	.109	.089	$Q_{\frac{5}{12},\frac{3}{4}}$	.0556	.0417	$Q_{\frac{5}{12},\frac{3}{4}}$	.0856	.0660

TABLE 2. Overview of regret points.

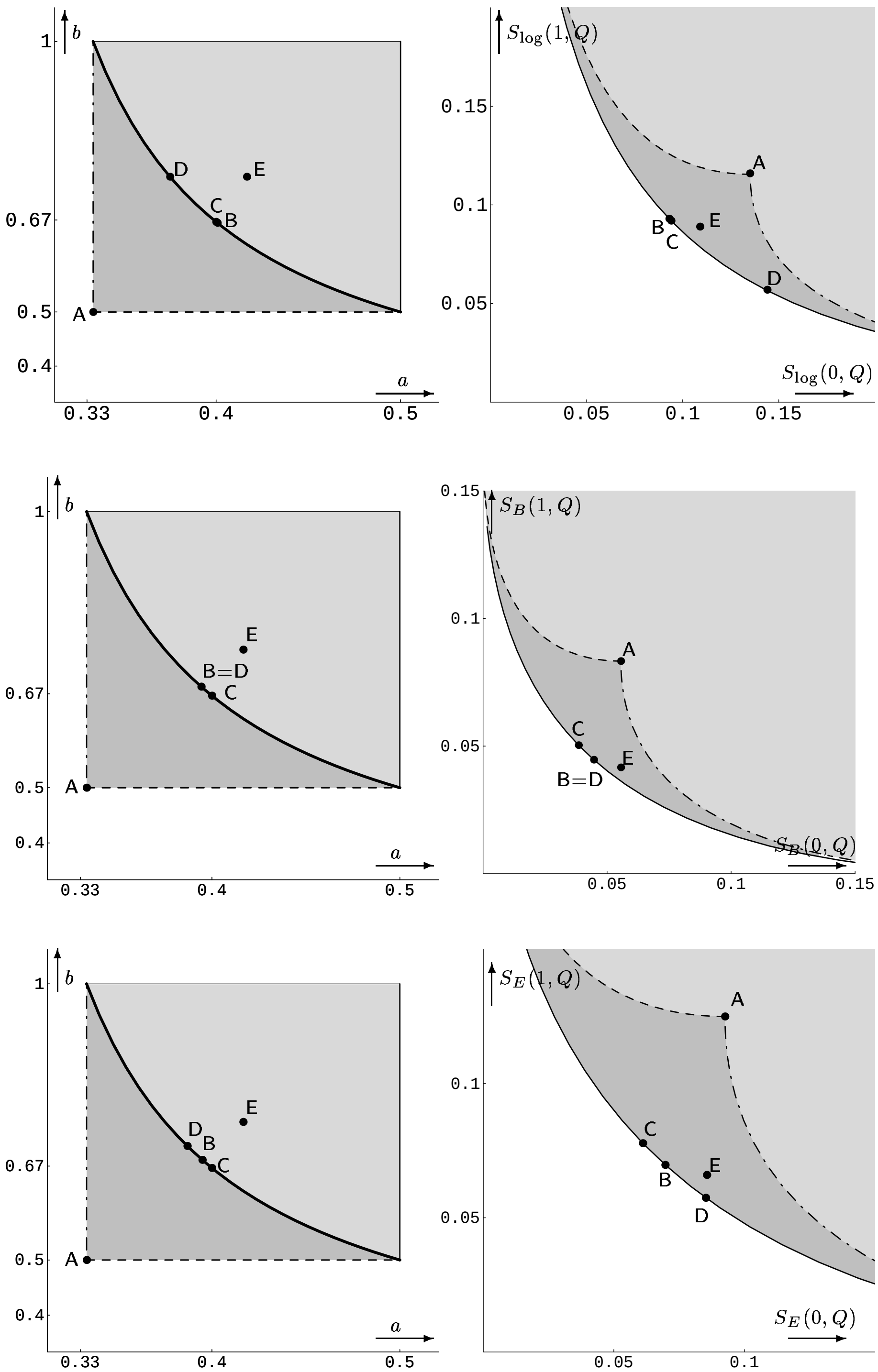

FIGURE 1. Visualization of procedures in the (a, b)-plane and the $(S(0, Q), S(1, Q))$-plane. From above to below: logarithmic, Brier and Epstein scoring rules. See the text for more details.

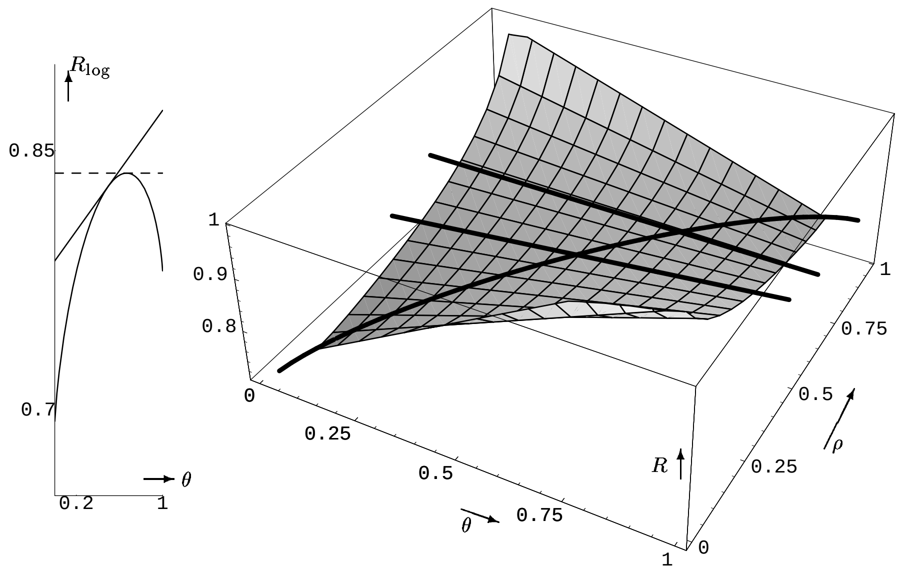

FIGURE 2. Risks for logarithmic loss.

risk $R^*_{\log}(\theta)$ (curve), the minimax risk procedure $R_{\log}(\theta, Q_{\frac{2}{3}})$ (dashed line) and the minimax regret procedure $R_{\log}(\theta, Q_{.4957})$ (solid line). In left of Figure 3 we see the minimax risk and minimax regret procedure (these figures also appeared in [1]). In the right part of Figure 2 the riskplane $R_{\log}(\theta, Q_\rho)$ $([\theta, \rho] \in (0,1)^2)$ is displayed. Notice that for fixed ρ, the risk is linear in θ. The three solid lines correspond to the envelope risk ($\theta = \rho$) and minimax risk ($\rho = \frac{2}{3}$) and minimax regret ($\rho = 0.4957$) procedures. The right part of Figure 3 shows a similar display, now in the regretplane. The figures on the left of Figure 2 can thus be described as the right parts viewed 'directly from the front'.

Conclusions. Of course, the procedures corresponding to A, C and E are independent of the choice of loss function, as long as this loss function is proper. Of course, the corresponding risks and regrets do shift. Note that in the case of Brier-loss, the values of the minimax risk and minimax regret actions are the same because the envelope risks in 0 and 1 are equal (this is coincidential). The position of the minimax risk and minimax regret procedures are different, but stay, of course, in the form Q_θ. The differences are also small, all procedures suggest a strategy Q_θ with θ close to $[\frac{1}{2}, \frac{2}{4}]$.

REFERENCES

1. C.J. Albers and W. Schaafsma, *How to assign probabilities if you must*, (to be published) (2000), 1.

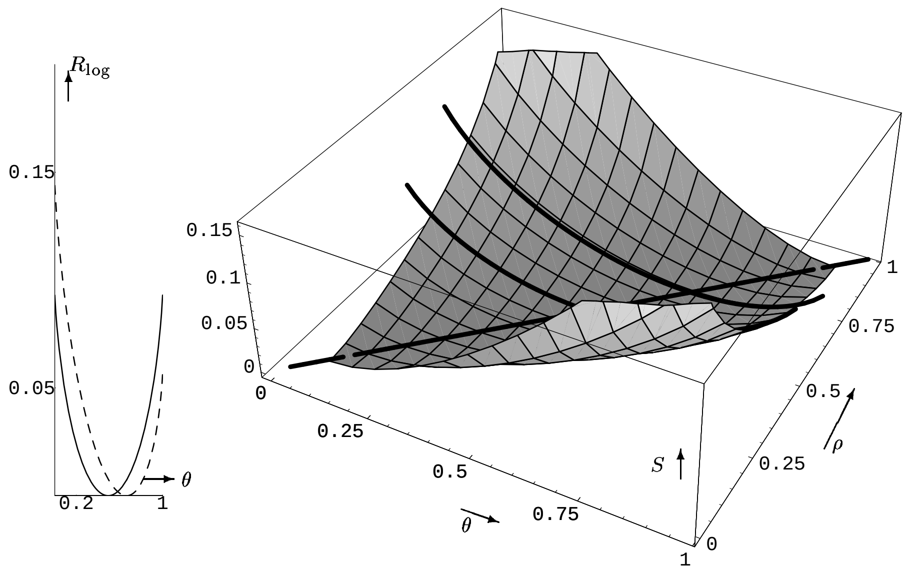

FIGURE 3. Regrets for logarithmic loss.